

## Time History Analysis Do accent last in word equation editor

Time history analysis is by Newmark B direct integration method.

Given the equations of motion for a lumped mass system:

$$M\ddot{u}_t + C\dot{u}_t + Ku_t = F_t$$

Where  $u_t$  is the displacement relative to the fixed supports at time  $t$  and  $F_t$  is a force vector of arbitrary time varying loads

For a structure uniformly subjected to ground acceleration  $\ddot{u}_g$ , the equations become:

$$M\ddot{u}_{rt} + C\dot{u}_{rt} + Ku_{rt} = -M\ddot{u}_{gt}$$

Where  $u_{rt}$  is the relative displacement of the structure with respect to ground or  $u_{rt} = u_t - u_{gt}$

For a structure subjected to non uniform ground acceleration, the contribution of acceleration from each support group is factored by multiplying the acceleration of that group by the deformation of each degree of freedom by the static unit deformation of the support group.

The equations of motion are solved using the Taylor series which is:

$$X(t+h) = X(t) + h \left(\frac{d}{dt}\right) X(t) + \frac{1}{2!} h^2 \left(\frac{d}{dt}\right)^2 X(t) + \frac{1}{3!} h^3 \left(\frac{d}{dt}\right)^3 X(t) + \dots \text{etc.}$$

For a given function, the Taylor series provides the exact answer if all the derivatives are found and all the terms are carried.

$$u_t = u_{t-\Delta t} + \Delta t \dot{u}_{t-\Delta t} + \frac{\Delta t^2}{2} \ddot{u}_{t-\Delta t} + \frac{\Delta t^3}{6} \dddot{u}_{t-\Delta t} + \dots$$

$$\dot{u}_t = \dot{u}_{t-\Delta t} + \Delta t \ddot{u}_{t-\Delta t} + \frac{\Delta t^2}{2} \dddot{u}_{t-\Delta t} + \dots$$

For the Newmark method, the series is truncated to:

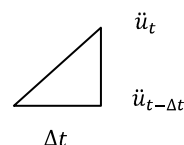
$$(1) \quad u_t = u_{t-\Delta t} + \Delta t \dot{u}_{t-\Delta t} + \frac{\Delta t^2}{2} \ddot{u}_{t-\Delta t} + \beta \Delta t^3 \ddot{u}_{t-\Delta t}$$

$$(2) \quad \dot{u}_t = \dot{u}_{t-\Delta t} + \Delta t \ddot{u}_{t-\Delta t} + \frac{\Delta t^2}{2} \dddot{u}_{t-\Delta t} + \gamma \Delta t^2 \ddot{u}_{t-\Delta t}$$

Where  $\beta$  and  $\gamma$  are values specified by the analyst. Typical values  $\beta = 0.25$  and  $\gamma = 0.50$  work for most cases.

The acceleration change between time steps is considered to be linear thus:

$$(3) \quad \ddot{u} = \frac{(\ddot{u}_t - \ddot{u}_{t-\Delta t})}{\Delta t} \quad \text{Note: Specify the script first, then the accent}$$



Substituting Equation 3 into 1

$$u_t = u_{t-\Delta t} + \Delta t \dot{u}_{t-\Delta t} + \frac{\Delta t^2}{2} \ddot{u}_{t-\Delta t} + \frac{\beta \Delta t^3 \ddot{u}_t}{\Delta t} - \frac{\beta \Delta t^3 \ddot{u}_{t-\Delta t}}{\Delta t}$$

$$u_t = u_{t-\Delta t} + \Delta t \dot{u}_{t-\Delta t} + \left(\frac{1}{2} - \beta\right) \Delta t^2 \ddot{u}_{t-\Delta t} + \beta \Delta t^2 \ddot{u}_t$$

Solve for  $\ddot{u}_t$

$$\ddot{u}_t = \frac{u_t - u_{t-\Delta t} - \Delta t \dot{u}_{t-\Delta t} + \left(\beta - \frac{1}{2}\right) \Delta t^2 \ddot{u}_{t-\Delta t}}{\beta \Delta t^2}$$

$$(4) \quad \ddot{u}_t = \frac{1}{\beta \Delta t^2} (u_t - u_{t-\Delta t}) - \left(\frac{1}{\beta \Delta t}\right) \dot{u}_{t-\Delta t} + \frac{\left(\beta - \frac{1}{2}\right)}{\beta} \ddot{u}_{t-\Delta t}$$

Substitute (3) into (2)

$$\dot{u}_t = \dot{u}_{t-\Delta t} + \Delta t \ddot{u}_{t-\Delta t} + \frac{\gamma \Delta t^2 \ddot{u}_t}{\Delta t} - \frac{\gamma \Delta t^2 \ddot{u}_{t-\Delta t}}{\Delta t}$$

$$(5) \quad \dot{u}_t = \dot{u}_{t-\Delta t} + (1 - \gamma) \Delta t \ddot{u}_{t-\Delta t} + \gamma \Delta t \ddot{u}_t$$

Substitute (4) into (5)

$$\begin{aligned} \dot{u}_t &= \dot{u}_{t-\Delta t} + (1 - \gamma) \Delta t \ddot{u}_{t-\Delta t} + \gamma \Delta t \left[ \frac{1}{\beta \Delta t^2} (u_t - u_{t-\Delta t}) - \left(\frac{1}{\beta \Delta t}\right) \dot{u}_{t-\Delta t} + \frac{\left(\beta - \frac{1}{2}\right)}{\beta} \ddot{u}_{t-\Delta t} \right] \\ &= \dot{u}_{t-\Delta t} + (1 - \gamma) \Delta t \ddot{u}_{t-\Delta t} + \frac{\gamma}{\beta \Delta t} (u_t - u_{t-\Delta t}) - \frac{\gamma}{\beta} \dot{u}_{t-\Delta t} + \frac{\gamma \Delta t}{\beta} \left(\beta - \frac{1}{2}\right) \ddot{u}_{t-\Delta t} \\ &= \frac{\gamma}{\beta \Delta t} (u_t - u_{t-\Delta t}) - \frac{\gamma}{\beta} u_{t-\Delta t} + \dot{u}_{t-\Delta t} + (1 - \gamma) \Delta t \ddot{u}_{t-\Delta t} + \frac{\gamma \Delta t}{\beta} \left(\beta - \frac{1}{2}\right) \ddot{u}_{t-\Delta t} \\ &= \frac{\gamma}{\beta \Delta t} (u_t - u_{t-\Delta t}) + \left(1 - \frac{\gamma}{\beta}\right) \dot{u}_{t-\Delta t} + (1 - \gamma) \Delta t \ddot{u}_{t-\Delta t} + \left(\gamma - \frac{\gamma}{2\beta}\right) \Delta t \ddot{u}_{t-\Delta t} \\ &= \frac{\gamma}{\beta \Delta t} (u_t - u_{t-\Delta t}) + \left(1 - \frac{\gamma}{\beta}\right) \dot{u}_{t-\Delta t} + \left[(1 - \gamma) + \left(\gamma - \frac{\gamma}{2\beta}\right)\right] \Delta t \ddot{u}_{t-\Delta t} \end{aligned}$$

$$(6) \quad \dot{u}_t = \frac{\gamma}{\beta \Delta t} (u_t - u_{t-\Delta t}) + \left(1 - \frac{\gamma}{\beta}\right) \dot{u}_{t-\Delta t} + \left(1 - \frac{\gamma}{2\beta}\right) \Delta t \ddot{u}_{t-\Delta t}$$

Rewrite (4) and (6) as

$$(7) \ddot{u}_t = b_1(u_t - u_{t-\Delta t}) + b_2\dot{u}_{t-\Delta t} + b_3\ddot{u}_{t-\Delta t}$$

$$(8) \dot{u}_t = b_4(u_t - u_{t-\Delta t}) + b_5\dot{u}_{t-\Delta t} + b_6\ddot{u}_{t-\Delta t}$$

Where:

$$b_1 = \frac{1}{\beta\Delta t^2}$$

$$b_2 = \frac{-1}{\beta\Delta t}$$

$$b_3 = \frac{(\beta - \frac{1}{2})}{\beta}$$

$$b_4 = \frac{\gamma}{\beta\Delta t}$$

$$b_5 = \left(1 - \frac{\gamma}{\beta}\right)$$

$$b_6 = \left(1 - \frac{\gamma}{2\beta}\right)\Delta t$$

Substitute (7) and (8) into equation of motion

$$[b_1(u_t - u_{t-\Delta t}) + b_2\dot{u}_{t-\Delta t} + b_3\ddot{u}_{t-\Delta t}]M + [b_4(u_t - u_{t-\Delta t}) + b_5\dot{u}_{t-\Delta t} + b_6\ddot{u}_{t-\Delta t}]C + Ku_t = F(t)$$

Collect  $u_t$  terms on the left and  $u_{t-\Delta t}$  on the other

$$(b_1M + b_4C + K)u_t = F_t + M(b_1u_{t-\Delta t} - b_2\dot{u}_{t-\Delta t} - b_3\ddot{u}_{t-\Delta t}) + C(b_4u_{t-\Delta t} - b_5\dot{u}_{t-\Delta t} - b_6\ddot{u}_{t-\Delta t})$$

The effective stiffness is

$$\bar{K} = (b_1M + b_4C + K)$$

The effective force vector is

$$\bar{F}_t = F_t + M(b_1u_{t-\Delta t} - b_2\dot{u}_{t-\Delta t} - b_3\ddot{u}_{t-\Delta t}) + C(b_4u_{t-\Delta t} - b_5\dot{u}_{t-\Delta t} - b_6\ddot{u}_{t-\Delta t})$$

Integration Steps

Step 1) Calculate terms  $b_1$  thru  $b_6$ ,  $\bar{K}$  and  $\bar{K}^{-1}$  These values do not change for linear elastic systems.

Step 2) Define initial displacement, velocities and accelerations

Step 3) Calculate load vector  $\bar{F}_t$

Step 4) Calculate  $u_t = \bar{K}^{-1} \bar{F}_t$

Step 5) Calculate  $\dot{u}_t$  vector using equation (8)

Step 6) Calculate  $\ddot{u}_t$  vector using equation (7)

Step 7) Calculate the next time value  $t = t + \Delta t$  and repeat Step 3 to 7

## Response Spectra Analysis

The equations of motion for an undamped lumped mass system, (ref. John Biggs Introduction to Structural Dynamics Appendix) are written in matrix form as:

$$[B]\{\ddot{y}\} + [K]\{y\} = \{F(t)\}$$

Where:

[B] = diagonal matrix containing masses of the system

{y} = column matrix of displacements

{ $\ddot{y}$ } = column matrix of accelerations

[K] = stiffness matrix

{F(t)} = applied dynamic force vector

If the system is vibrating in a normal mode (i.e., free not forced vibration), we may make the substitutions

$$\{y\} = \{a_n\} \sin \omega_n t$$

$$\{\ddot{y}\} = -\omega_n^2 \{a_n\} \sin \omega_n t$$

$$\{F(t)\} = 0$$

where  $\{a_n\}$  is the vector of modal displacements of the  $n^{\text{th}}$  mode (eigenvector).

$\omega_n$  is the rotational frequency of mode  $n$ .

to obtain:

$$-\omega_n^2[B]\{a_n\} + [K]\{a_n\} = 0$$

or

$$[K]\{a_n\} = \omega_n^2[B]\{a_n\}$$

## Eigenvectors and Eigen Value extraction

The equation  $[K]\{a_n\} = \omega_n^2[B]\{a_n\}$  represents the equilibrium condition for the free response of mode,  $n$ .

The equation  $KM = \lambda BM$  represents the entire system of all modes and eigenvalues where:

$K$  = Stiffness Matrix  $N_{\text{dof}} \times N_{\text{dof}}$

$N_{\text{dof}}$  is the number of degrees of freedom)

$M$  = Mode shape matrix  $N_{\text{dof}} \times N_{\text{dof}}$  containing  $\{a_n\}$  vectors. (Columns are mode shapes)

$B$  = Mass Diagonal Matrix  $N_{\text{dof}} \times N_{\text{dof}}$ .

$\lambda$  = Diagonal Matrix containing all Eigenvalues

This equation is converted to a more suitable form for solving. (Ref: Crandell equations 2-176, 2-177 and 2-179)

$$AC = \lambda C$$

Where:

$$A = B^{-1/2}KB^{-1/2}$$

$$C = B^{1/2}M$$

This results in a symmetrical  $A$  matrix. Symmetry is necessary for the chosen Eigen extractor.

The eigenvalues and vectors are extracted by reducing A to tri-diagonal form using Householder reduction and QL, utilizing public routines from EISPACK which were rewritten into VBA.

The real mode matrix is then obtained by  $M = B^{-1/2}C$

Rotational mass is insignificant for piping and beam elements and could be ignored, however a zero value is not tolerated by the Eigen extractor thus the program estimates a value.

To increase the speed of the extraction, the equations are consolidated, removing rows and columns associated with rigidly supported degrees of freedom.

Comment: The Eigen extraction used here is not fast because all modes are found, regardless of the cut off frequency specified by the user. However it is very reliable. When this program was benchmarked against other industry programs, it was found that some other programs completely missed some symmetrical modes of the same frequency.

### **Response Spectra Analysis**

Note that the program normalizes mode shapes such that  $\{a_n\}^T B \{a_n\} = 1.0$ .

All modes are orthogonal to each other such that  $\{a_k\}^T \{a_j\} = 0, j \neq k$

Given normalized mode shapes as indicated above, the mass participation factors are calculated as:

$$MPF = M^T B R$$

(see “matrix calc of MPF Spectra Test 27.xls” in qa folder.)

Where:

R is a matrix containing column r vectors resulting from displacing all supported degrees of freedom associated with a spectra support group, one unit in translational direction d and solving statically for the deformations of the non supported degrees of freedom. Dimensions are Ndof rows by 3Nsg columns where Nsg are the number of spectra support groups. If all supports are defined as one spectra group then the R matrix defaults to 3 columns, one for each of the translational directions, with a 1.0 in each degree of freedom that matches the direction of the displacement which is equivalent to the Kronecker Delta referenced in equation A.2 in NRC 1.92 Rev 2.

MPF is the mass participation factor Matrix. The rows correspond to the associated mode and the columns to the corresponding r vector which indicates the direction and spectra support group. The rotational participation factors are not utilized. Dimensions are Ndof rows by 3Nsg

columns where Nsg are the number of spectra support groups. If all supports are defined as one spectra group then the MPF matrix defaults to 3 columns, one for each direction.

Note when there are more than one spectra support group the algebraic combination of the mass participation factors for each level results in single spectra support group.

$$MPF_{n,d} = \sum_L MPF_{n,d,L} \text{ where } L \text{ is from } 1 \text{ to the number of spectra support groups.}$$

This can be shown by rewriting this equation as.

$$MPF_{n,d} = \sum_L \sum_i a_{n,i} B_i r_{i,l,d} \text{ , } L=1, \text{Nspectra Groups, } i=1 \text{ to } N_{dof}$$

$$= \sum_i (a_{n,i} B_i \sum_L r_{i,l,d})$$

and since  $\sum_L r_{i,l,d} = d_{i,D}$  then

$= \sum_i (a_{n,i} B_i d_{i,D})$  which is an equivalent definition of Mass Participation for a single response spectra support group. (See NRC 1.92 Rev 2 and note  $\{a_n\}^T B \{a_n\} = 1.0$ )

$a_{n,i}$  is the eigenvector value for degree of freedom  $i$ , mode  $n$ .

$B_i$  is the lumped mass for dof  $i$

$r_{i,l,d}$  is the value of the R matrix for dof  $i$ , direction  $d$  and Spectra Group  $L$ .

$d_{i,D}$  is the Kronecker Delta referenced in NRC 1.92 Rev 2 which is equal to 1.0 when DOF  $i$  is in the same direction as  $D$  and 0 if not or if the DOF is a rotation.

The modal responses are specified as;

$$U_{i,n,L,d} = S_{n,L,d} \text{ mpf}_{n,L,d} a_{n,i}$$

Where:

$mpf_{n,L,d}$  is the value of the MPF matrix for node n, direction d of input motion and spectra support group L.

$U_{i,n,L,d}$  is the displacement response for degree of freedom i, mode n, direction d and spectra support group L.

$S_{n,L,d}$  is the response spectra value in displacement for mode n, direction d and spectra support group L. Spectra that are input in velocity or acceleration are converted to displacement by  $S=SVel/\omega_n$  and  $S=SAccel/\omega_n^2$ .

Note: Since  $MPF_{n,d} = \sum_L MPF_{n,d,L}$

Then the algebraic combination of the modal responses of all the spectra support groups results in the simple case:

$$U_{n,,d,i} = \sum_L S_{n,L,d} MPF_{n,L,d} a_{n,i}$$

which is helpful information for QA testing purposes.

### **Combination order of responses from modes, directions and spectra groups**

Displacements are calculated for each mode due to each defined response spectra. The program uses these displacements to calculate the equilibrium forces and moments for each mode up to and including modes matching the user defined cut off frequency. If the user chooses to combine modal responses last, then the program combines each set of responses from a given mode from each spectra first, before combining modes. If modes are combined first than the resulting components for each spectra are combined later. The combination of the spatial and support group spectra components are by SRSS or ABS as defined by the user. All Modal Combinations options presented in NRC 1.92 Rev 1 and Rev 2 are available.

### **Combination Order Options**

Given closely spaced modes, the order of combination of spatial, spectra support groups and modal components could make a significant difference. Combining the responses of individual modes due to different response spectra first can be more conservative. This is because some closely spaced modes maybe tied to different directions as indicated by the Mass Participation factor. For example given two closely spaced modes where mode 1 has significant response in the X direction only and mode 2 having response predominately in the Y direction, if the modal response were combined first then there is little or no additional calculated response if modes are combined by accepted NRC methods for closely

spaced modes. If however the response from mode 1 due to each spectra are first combined into one component and likewise for mode 2, then combining the resulting components from Mode 1 and 2 using a closely spaced mode method from NRC 1.92 Rev 1 would result in an absolute sum combination. (Using Rev 2 could result in a less conservative combination than Rev 1 however would still be potentially more conservative than if the modes were combined before the special components.) The less conservative combination order method maybe more appropriate if in fact the spatial components in the X and Y direction are independent, or in other words the time histories that produced the spectra are unrelated in time, which significantly decreases the chances of these two modes hitting peaks at the same time. If the spatial components are dependent, then the peaks caused by these modes from different direction spectra are more likely to peak at the same time, thus combining modes first maybe non-conservative.

Similarly, when combining responses from spectra specific to a subset of support groups (sometimes called levels). In most cases, the different support group spectra for a given direction will be defined by the same time history for a given direction. In other words, if each support group defined for a given direction is time dependant with all other levels for that direction then they should be combined before the modal responses are combined.

Note if all supports are in the same spectra group, then the program defaults to a single spectra support group thus there will only be set of modal responses calculated from each of the 3 directions.

There are a number of options for combination order. They include DLM (Default) which for each mode all the responses from all spectra are combined first into one set of responses for each mode. (The D representing Direction X, Y, Z Spectra and L representing Support Group Spectra, sometimes call Levels and M modes). The remaining modal components are then combined based on the modal combination method specified by the user. Note there are some modal combinations that could be overly conservative with this option since with this method, the sign information at the model level is lost. Specifically 1.92 Rev 2 combinations using the Rosenblueth or the Der Kiureghian Correlation Coefficients would be conservative since it utilizes the equivalent of algebraic summation of products of correlated modes. (Note that the earlier version of Rosenblueth in 1.92 Rev 1 specifies absolute sum of the products of correlated modes but Rev 2 does not.) In addition, the Gupta and Lindley-Yow Methods, (also available in JoePipe) which address modes with both periodic and rigid response would be conservative for the DLM method for the same reason since the rigid components of some modes are combined algebraically. The DLM is faster than the other combination order methods since less computation is required.

Other combination order methods are LMD, MLD, DML

### **Modal Combination Options**

All Modal Combinations options presented in NRC 1.92 Rev 1 and Rev 2 are available, including CQC methods utilizing Rosenblueth and Der Kiureghian Correlation Coefficients. Both Gupta and Lindley-Yow Methods are available for processing Modes with Both Periodic and Rigid Response

Components. Rigid Response components of such modes are algebraically combined with missing mass responses. The combined Rigid Response components of modes and missing mass responses are combined with the Modal Periodic combinations with SRSS, as if it were an additional widely spaced mode. In addition, the user can specify algebraic combination of modes, spatial and support group components as well as rigid range responses for testing and research purposes, if also using the MLD option for combination order.

## Rigid range correction

It is typical not to process modes in the rigid range of a response spectrum. Without evaluation of the rigid modes, some responses, particularly support loads, will be non conservative. Even if all the modes are found, a mass that is lumped at a supported degree of freedom cannot be accounted for since the value of all the mode shapes are zero. The Rigid range correction, accounts for this as well as the missing modes thru a static analysis which utilizes the calculated missing mass and the rigid range acceleration.

Calculation of Captured Mass

$$\text{CapturedMassRatio}_{i,L,d} = \sum_n \text{MPF}_{n,d,L} \mathbf{a}_{n,i} \quad \text{where } n=1 \text{ to } \text{Nmodes} \text{ (Number of modes)}$$

$$\text{MissMass}_{i,L,d} = \mathbf{b}_i \left( \mathbf{r}_{i,L,d} - \text{CapturedMassRatio}_{L,d,i} \right)$$

$$\mathbf{F}_{i,L,d} = \text{ZPA}_{n,d} \text{MissMass}_{i,L,d}$$

Where  $\mathbf{F}_{i,L,d}$  is the calculated inertial force for DOF  $i$ , Direction  $d$  and support group  $L$ . With a set of inertial forces, static equilibrium is calculated for each support group and direction. If the user has specified Gupta or Lindley-Yow, the resulting displacement, support reactions and member forces are combined with rigid components of modes with algebraic combination. These results are then combined with periodic modal responses using SRSS. The user can also specify Algebraic for QA or research purposes.

For Single Support Group response spectra, the equations default to the equivalent of the NRC 1.92 Appedix A method:

$$\text{MissMass}_{d,i} = \mathbf{b}_i \left( \mathbf{d}_{i,d} - \text{CapturedMassRatio}_{d,i} \right)$$

$$\mathbf{F}_{i,d} = \text{ZPA}_d \text{MissMass}_{d,i}$$

$\mathbf{d}_{i,d}$  is the Kronecker Delta referenced in NRC 1.92 Rev 2 which is equal to 1.0 when the DOF is in the same direction as  $d$  and 0 if not or if the DOF is a rotation.

i and other subscripts are defined previously

The program provides the user with the missing mass inertia forces that was used in the analysis for each degree of freedom, for each direction and for each response spectra support group. The program also provides the user with the sum total of captured and missing mass for comparison with the total amount of mass in the model.

The output provides the user with the algebraic summation of the captured and missing mass. In addition, the summation of the product of the mass and  $d_{i,d}$  ( the Kronecker Delta) is also provided, since this is required for agreement with summation of the square of the mass participation factors of the modes considered.

Calculation of summation of mass for output:

Warning, the definition of modal mass here is only used for output purposes. It would be incorrect to use it for the rigid range correction calculation.

The total captured mass for a given mode and given spectra support group is:

$$\text{ModalMass}_{n,L,d} = \sum_i \text{MPF}_{n,d,L} a_{n,i} b_i d_{i,d} \text{ ( Used for Output Purposes Only)}$$

The total mass for a given mode for all support groups:

$$\begin{aligned} \text{ModalMass}_{n,d} &= \sum_L \sum_i \text{MPF}_{n,d,L} a_{n,i} b_i d_{i,d} \\ &= \sum_i (a_{n,i} b_i d_{i,d} \sum_L \text{MPF}_{n,d,L}) \\ &= \text{MPF}_{n,d} \sum_i a_{n,i} b_i d_{i,d} \quad (\text{since } \text{MPF}_{n,d} = \sum_L \text{MPF}_{n,d,L} ) \\ \text{ModalMass}_{n,d} &= (\text{MPF}_{n,d})^2 \end{aligned}$$

Missing nodal accelerations are also calculated by dividing the missing nodal inertial forces by the nodal mass.

Displacements and resulting equilibrium support forces and element forces are calculated using the inertial forces and then combined with other modes utilizing SRSS or (algebraic if utilizing a QA test option.)

The rigid range acceleration or ZPA is obtained from the input spectra. The program uses the value obtained from the highest mode processed. If there is a higher spectra value above the highest processed mode, it is used instead.